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Laguerre Polynomials Generalized to a Certain Discrete Sobolev Inner Product Space

H. G. MEIJER

Delft University of Technology, Faculty of Technical Mathematics and Informatics, Mekelweg 4, 2628 CD Delft, The Netherlands

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We are concerned with the set of polynomials $\{S_n^{M,N}\}$ which are orthogonal with respect to the discrete Sobolev inner product

$$\langle f, g\rangle = \int_0^\infty w(x)\, f(x)\, g(x)\, dx + Mf(0)\, g(0) + Nf'(0)\, g'(0),$$

where w is a weight function, $M \geqslant 0$, $N \geqslant 0$. We show that these polynomials can be described as a linear combination of standard polynomials which are orthogonal with respect to the weight functions $w(x)$, $x^2w(x)$, and $x^4w(x)$. The location of the zeros of $S_n^{M,N}$ is given in relation to the position of the zeros of the standard polynomials.

1. INTRODUCTION

Several authors generalize the concept of standard orthogonal polynomials to orthogonal polynomials in a Sobolev inner product space. We mention here Althammer [1], Brenner [3], Cohen [5], and more recently Bavinck, Meijer [2], Koekoek [7], Marcellan, Ronveaux [10], and Iserles, Koch, Nørsett, Sanz-Serna [6].

In the present paper we investigate the polynomials $\{S_n^{M,N}\}$ which are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_0^\infty w(x)\, f(x)\, g(x)\, dx + Mf(0)\, g(0) + Nf'(0)\, g'(0),$$

where w is a weight function, $M \geqslant 0$, $N \geqslant 0$.

We show that these polynomials can be expressed as

$$S_n^{M,N}(x) = B_1 K_n(x) + B_2 x K_{n-1}^{(2)}(x) + B_3 x^2 K_{n-2}^{(4)}(x), \tag{1.1}$$

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where $\{K_n\}$, $\{K_n^{(2)}\}$ and $\{K_n^{(4)}\}$ are the sets of standard orthogonal polynomials ($M=N=0$) with respect to the weight functions $w(x)$, respectively, $x^2w(x)$ and $x^4w(x)$.

Furthermore we describe the position of the zeros of $S_n^{M,N}(x)$ in relation to the zeros of $K_n(x)$ and $K_{n-1}^{(2)}(x)$.

In Section 2 we recall some well-known results on the standard polynomials and derive some simple relations between K_n, $K_n^{(2)}$, and $K_n^{(4)}$. In Section 3 we define the polynomials $S_n^{M,N}$ and prove relation (1.1). In Section 4 the coefficients B_1, B_2, and B_3 in (1.1) are studied in more detail.

Section 5 contains some simple results on the zeros of $S_n^{M,N}$. Finally in Section 6 the location of the zeros of $S_n^{M,N}$ in relation to the zeros of K_n and $K_{n-1}^{(2)}$ is derived.

Some results in this paper are direct generalizations of results in [9], where the weight function is the Laguerre weight $w(x)=x^\alpha e^{-x}$ $(\alpha>-1)$; the results in Section 6, however, are completely new.

2. The Standard Polynomials

Let w denote a weight function on $(0,\infty)$, i.e., $w(x)\geq 0$, all moments

$$c_n=\int_0^\infty w(x)\,x^n\,dx, \qquad n=0,1,2,...$$

exist and $c_0\neq 0$.

The support of w, i.e., the closure of the set $\{x \mid w(x)>0\}$ may be a real subset of $[0,\infty)$; the point $x=0$ may be outside or on the boundary of the support of w.

Consider the inner product

$$(f,g)=\int_0^\infty w(x)\,f(x)\,g(x)\,dx. \tag{2.1}$$

Define the set of standard polynomials $\{K_n\}$ by

$$K_0(x)\equiv 1,$$

$$K_n(x)=\begin{vmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix} \qquad \text{for} \quad n\geq 1. \tag{2.2}$$

Then we have, for $0 \leqslant i \leqslant n-1$, that $(x^i, K_n) = 0$, showing that $\{K_n\}$ is an orthogonal set with respect to the inner product (2.1).

Let a_n denote the leading coefficient of $K_n(x)$, then we have

$$(x^n, K_n) = a_{n+1} \qquad \text{for} \quad n \geqslant 0.$$

Then $(a_n x^n, K_n) = a_n a_{n+1}$. On the other hand, $a_n x^n = K_n(x) + p_{n-1}(x)$, for some polynomial p_{n-1} of degree $\leqslant n-1$. Then $(a_n x^n, K_n) = (K_n, K_n) > 0$. This implies $a_n a_{n+1} > 0$ for $n \geqslant 0$. Since $a_0 = 1$ all leading coefficients a_n are positive.

In the same way we can describe the sets of polynomials $\{K_n^{(2)}\}$ and $\{K_n^{(4)}\}$ which are orthogonal with respect to the weight functions $w(x)x^2$ and $w(x)x^4$ respectively. They are defined for $n \geqslant 1$ by the determinant (2.2) with c_i replaced by c_{i+2}, respectively c_{i+4}. For $n = 0$ we define $K_0^{(2)}(x) = K_0^{(4)}(x) \equiv 1$.

We will often use the following result: all zeros of K_n, $K_n^{(2)}$ and $K_n^{(4)}$ are real, simple and lie in $(0, \infty)$.

Especially this implies, since the leading coefficients are positive,

$$\operatorname{sgn} K_n(0) = \operatorname{sgn} K_n^{(2)}(0) = \operatorname{sgn} K_n^{(4)}(0) = (-1)^n \qquad \text{for} \quad n \geqslant 0, \tag{2.3}$$

$$\operatorname{sgn} K_n'(0) = \operatorname{sgn} K_n^{(2)\prime}(0) = \operatorname{sgn} K_n^{(4)\prime}(0) = (-1)^{n-1} \qquad \text{for} \quad n \geqslant 1. \tag{2.4}$$

LEMMA 2.1. *The following relations hold*:

$$\int_0^\infty w(x)\, K_n^{(2)}(x)\, dx = K_{n+1}'(0) \qquad \textit{for} \quad n \geqslant 0. \tag{2.5}$$

$$\int_0^\infty w(x) x\, K_n^{(2)}(x)\, dx = -K_{n+1}(0) \qquad \textit{for} \quad n \geqslant 0. \tag{2.6}$$

$$\int_0^\infty w(x) x^2\, K_n^{(2)}(x)\, dx = \begin{cases} 0 & \textit{for} \quad n \geqslant 1 \\ c_2 & \textit{for} \quad n = 0. \end{cases} \tag{2.7}$$

$$\int_0^\infty w(x) x^2\, K_n^{(4)}(x)\, dx = K_{n+1}^{(2)\prime}(0) \qquad \textit{for} \quad n \geqslant 0. \tag{2.8}$$

$$\int_0^\infty w(x) x^3\, K_n^{(4)}(x)\, dx = -K_{n+1}^{(2)}(0) \qquad \textit{for} \quad n \geqslant 0. \tag{2.9}$$

Proof. For $n \geqslant 1$ relations (2.5) and (2.6) follow from the determinantal representation (2.2). Relation (2.7) for $n \geqslant 1$ is a direct consequence of the orthogonality. For $n = 0$ we observe $K_n^{(2)}(x) \equiv 1$ and $K_1(x) = c_0 x - c_1$ and (2.5), (2.6), and (2.7) follow for $n = 0$.

Relation (2.8) and (2.9) are a direct consequence of (2.5), respectively (2.6).

3. The Polynomials S_n

Consider the inner product

$$\langle f, g\rangle = \int_0^\infty w(x)\, f(x)\, g(x)\, dx + Mf(0)\, g(0) + Nf'(0)\, g'(0), \quad (3.1)$$

where $M \geqslant 0$, $N \geqslant 0$. Then

$$\langle x^i, x^j\rangle = c_{i+j} + \begin{cases} M & \text{if } i=j=0, \\ N & \text{if } i=j=1, \\ 0 & \text{otherwise.} \end{cases}$$

Define the set of polynomials $\{S_n^{M,N}\}$ by

$$S_0^{M,N}(x) \equiv 1, \; S_1^{M,N}(x) = \begin{vmatrix} c_0+M & c_1 \\ 1 & x \end{vmatrix},$$

$$S_n^{M,N}(x) = \begin{vmatrix} c_0+M & c_1 & c_2 & \cdots & c_n \\ c_1 & c_2+N & c_3 & \cdots & c_{n+1} \\ c_2 & c_3 & c_4 & \cdots & c_{n+2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_n & c_{n+1} & \cdots & c_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{vmatrix} \quad \text{for} \quad n \geqslant 2. \quad (3.2)$$

We often write $S_n(x)$ instead of $S_n^{M,N}(x)$. For $0 \leqslant i \leqslant n-1$ we have $\langle x^i, S_n\rangle = 0$. Let $\tilde{a}_n$ denote the leading coefficient of S_n, then for $n \geqslant 2$

$$\langle x^n, S_n\rangle = \tilde{a}_{n+1}.$$

For $n=0$ we obtain $\langle 1, S_0\rangle = c_0 + M = \tilde{a}_1$ and for $n=1$,

$$\langle x, S_1\rangle = \begin{vmatrix} c_0+M & c_1 \\ c_1 & c_2 \end{vmatrix} + N(c_0+M) = \tilde{a}_2.$$

Hence $\langle x^n, S_n\rangle = \tilde{a}_{n+1}$ for $n \geqslant 0$. Then $0 < \langle S_n, S_n\rangle = \tilde{a}_n \langle x^n, S_n\rangle = \tilde{a}_n \tilde{a}_{n+1}$ for $n \geqslant 0$. Since $\tilde{a}_0 = 1$, all leading coefficients are positive. We have found that $\{S_n\}$ is a set of orthogonal polynomials on $(0, \infty)$ with respect to the inner product (3.1).

Evaluating the determinant (3.2) it follows that S_n can be written as $S_n(x)=K_n(x)+MxK_{n-1}^{(2)}(x)+MNx^2K_{n-2}^{(4)}(x)+Nq_n(x)$, for some polynomial q_n of degree $\leqslant n$. Here the polynomials $K_n(x)$, $xK_{n-1}^{(2)}(x)$, and $x^2K_{n-2}^{(4)}(x)$ occur. We observe that for $2\leqslant i\leqslant n-1$,

$$\langle x^i, K_n(x)\rangle = 0,$$

$$\langle x^i, xK_{n-1}^{(2)}(x)\rangle = \int_0^\infty w(x)\, x^2 x^{i-1} K_{n-1}^{(2)}(x)\, dx = 0,$$

$$\langle x^i, x^2K_{n-2}^{(4)}(x)\rangle = \int_0^\infty w(x)\, x^4 x^{i-2} K_{n-2}^{(4)}(x)\, dx = 0.$$

Then S_n can be written as

$$S_n(x) = B_1 K_n(x) + B_2 xK_{n-1}^{(2)}(x) + B_3 x^2 K_{n-2}^{(4)}(x),$$

if the constants B_1, B_2, and B_3 are chosen in such a way that

$$\langle 1, S_n\rangle = 0 \qquad \text{for} \quad n\geqslant 1. \tag{3.3}$$

$$\langle x, S_n\rangle = 0 \qquad \text{for} \quad n\geqslant 2. \tag{3.4}$$

[As usual we define $K_{-1}^{(2)}(x)=K_{-1}^{(4)}(x)=K_{-2}^{(4)}(x)\equiv 0$.] Using (3.1), (2.6), (2.8), and (2.9) we conclude that B_1, B_2, and B_3 had to satisfy the equations

$$B_1 MK_n(0) - B_2 K_n(0) + B_3 K_{n-1}^{(2)\prime}(0) = 0 \qquad \text{for} \quad n\geqslant 1,$$

$$B_1 NK_n'(0) + B_2 NK_{n-1}^{(2)}(0) - B_3 K_{n-1}^{(2)}(0) = 0 \qquad \text{for} \quad n\geqslant 2.$$

We take

$$B_1 = 1 - \frac{K_{n-1}^{(2)\prime}(0)}{K_n(0)} N \qquad \text{for} \quad n\geqslant 0.$$

Then we find, using Cramer's rule for $n\geqslant 2$

$$B_2 = M + \frac{K_n'(0)}{K_n(0)} \frac{K_{n-1}^{(2)\prime}(0)}{K_{n-1}^{(2)}(0)} N \qquad \text{for} \quad n\geqslant 1,$$

$$B_3 = MN + \frac{K_n'(0)}{K_{n-1}^{(2)}(0)} N \qquad \text{for} \quad n\geqslant 2.$$

We have obtained the following result.

THEOREM 3.1. *For $n\geqslant 0$ the polynomial $S_n^{M,N}$ can be written as*

$$S_n^{M,N}(x) = B_1 K_n(x) + B_2 xK_{n-1}^{(2)}(x) + B_3 x^2 K_{n-2}^{(4)}(x),$$

where

$$B_1 = 1 - \alpha_n N, \qquad B_2 = M + \alpha_n \beta_n N, \qquad B_3 = MN + \beta_n N,$$

with

$$\alpha_n = \frac{K_{n-1}^{(2)}(0)}{K_n(0)} \quad \textit{for} \quad n \geqslant 0, \qquad \beta_n = \frac{K_n'(0)}{K_{n-1}^{(2)}(0)} \quad \textit{for} \quad n \geqslant 1. \tag{3.5}$$

Remark 3.1. It follows from (2.3) and (2.4) that $\alpha_n \geqslant 0$, $\beta_n > 0$, thus $B_2 \geqslant 0$, $B_3 \geqslant 0$. On the other hand, B_1 may be negative.

Remark 3.2. The theorem implies

$$S_n(0) = K_n(0)(1 - \alpha_n N), \tag{3.6}$$

$$S_n'(0) = K_n'(0) + MK_{n-1}^{(2)}(0). \tag{3.7}$$

Then $S_n(0)$ is independent of M and $S_n'(0)$ is independent of N. This can also directly be concluded from the determinant (3.2). We observe that $S_n'(0)$ always has the same sign as $K_n'(0)$, i..e, $(-1)^{n-1}$. On the other hand, $S_n(0)$ and $K_n(0)$ have different signs if $B_1 = 1 - \alpha_n N$ is negative. We will use this fact in the discussion on the zeros of S_n.

4. Monotonicity of α_n and β_n

In this section we prove that the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ defined in (3.5) are monotonic.

We start with the well-known relation of Christoffel–Darboux for the polynomials $\{K_n^{(2)}\}$,

$$(x-u) \sum_{i=0}^{n} \frac{K_i^{(2)}(x)\, K_i^{(2)}(u)}{b_i b_{i+1}} = \frac{1}{b_{n+1}^2} \{K_{n+1}^{(2)}(x)\, K_n^{(2)}(u) - K_n^{(2)}(x)\, K_{n+1}^{(2)}(u)\}, \tag{4.1}$$

where b_n denotes the leading coefficient of $K_n^{(2)}(x)$.

Theorem 4.1. *The sequence $\{\beta_n\}$ is decreasing.*

Proof. Multiply (4.1) by $w(u)$ and integrate over $(0, \infty)$. Then (2.5) and (2.6) give

$$x \sum_{i=0}^{n} \frac{K_i^{(2)}(x)}{b_i b_{i+1}} K_{i+1}'(0) + \sum_{i=0}^{n} \frac{K_i^{(2)}(x)}{b_i b_{i+1}} K_{i+1}(0)$$
$$= \frac{1}{b_{n+1}^2} \{K_{n+1}^{(2)}(x)\, K_{n+1}'(0) - K_n^{(2)}(x)\, K_{n+2}'(0)\}.$$

For $x=0$ we obtain

$$\frac{1}{b_{n+1}^2}\{K_{n+1}^{(2)}(0)\,K'_{n+1}(0)-K_n^{(2)}(0)\,K'_{n+2}(0)\}=\sum_{i=0}^{n}\frac{K_i^{(2)}(0)\,K_{i+1}(0)}{b_i b_{i+1}}.$$

The right hand term is negative by (2.3). Since $K_n^{(2)}(0)\,K_{n+1}^{(2)}(0)$ is negative too, this implies

$$\beta_{n+1}=\frac{K'_{n+1}(0)}{K_n^{(2)}(0)}>\frac{K'_{n+2}(0)}{K_{n+1}^{(2)}(0)}=\beta_{n+2}.$$

THEOREM 4.2. *The sequence* $\{\alpha_n\}$ *is increasing.*

Proof. We now multiply (4.1) by $uw(u)$ and integrate over $(0,\infty)$. With (2.6) and (2.7) we obtain

$$-x\sum_{i=0}^{n}\frac{K_i^{(2)}(x)}{b_i b_{i+1}}K_{i+1}(0)-1$$
$$=\frac{1}{b_{n+1}^2}\{-K_{n+1}^{(2)}(x)\,K_{n+1}(0)+K_n^{(2)}(x)\,K_{n+2}(0)\}.$$

Differentiating this relation and substituting $x=0$ we obtain

$$-\sum_{i=0}^{n}\frac{K_i^{(2)}(0)\,K_{i+1}(0)}{b_i b_{i+1}}=\frac{1}{b_{n+1}^2}\{-K_{n+1}^{(2)\prime}(0)\,K_{n+1}(0)+K_n^{(2)\prime}(0)\,K_{n+2}(0)\}.$$

The left hand side is positive. Since $K_{n+1}(0)\,K_{n+2}(0)$ is negative this implies

$$\alpha_{n+2}=\frac{K_{n+1}^{(2)\prime}(0)}{K_{n+2}(0)}>\frac{K_n^{(2)\prime}(0)}{K_{n+1}(0)}=\alpha_{n+1}.$$

COROLLARY 4.1. *In view of* (3.6), *Theorem* 4.2 *implies: if* $S_n(0)$ *and* $K_n(0)$ *have different signs for* $n=n_0$, *then they have different signs for all* n *with* $n>n_0$. *Remember* $\operatorname{sgn} K_n(0)=(-1)^n$.

THEOREM 4.3. *The sequence* $\{\alpha_n\beta_n\}$ *is increasing.*

Proof. By (3.5) we have

$$\alpha_n\beta_n=\frac{K_n'(0)\,K_{n-1}^{(2)\prime}(0)}{K_n(0)\,K_{n-1}^{(2)}(0)}\qquad\text{for}\quad n\geqslant 1.$$

Let $x_1<x_2<\cdots<x_n$ denote the zeros of K_n. Then

$$\left|\frac{K_n'(0)}{K_n(0)}\right|=\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}.\tag{4.2}$$

Let $\eta_1 < \eta_2 < \cdots < \eta_{n+1}$ denote the zeros of K_{n+1}. It is well known that the zeros of K_n and K_{n+1} mutually separate each other, i.e.,

$$\eta_i < x_i < \eta_{i+1} \qquad \text{for} \quad i = 1, 2, ..., n.$$

Hence

$$\left|\frac{K'_{n+1}(0)}{K_{n+1}(0)}\right| = \frac{1}{\eta_1} + \cdots + \frac{1}{\eta_{n+1}} > \frac{1}{\eta_1} + \cdots + \frac{1}{\eta_n} > \frac{1}{x_1} + \cdots + \frac{1}{x_n} = \left|\frac{K'_n(0)}{K_n(0)}\right|. \tag{4.3}$$

The same result holds for $K_n^{(2)}$. This implies that $\{\alpha_n \beta_n\}$ is increasing.

Remark 4.1. Suppose that the weight function w is such that $|K'_n(0)/K_n(0)| \to \infty$ if $n \to \infty$. It follows from the proof of Theorem 4.3 that $\alpha_n \beta_n \to \infty$. Since, by Theorem 4.1, $\{\beta_n\}$ is decreasing we have $\alpha_n \to \infty$. Then (3.6) implies that $S_n(0)$ and $K_n(0)$ have different signs if n is sufficiently large.

We mention two situations in which this condition on the weight function w is satisfied.

A. Suppose that the support of w is contained in the finite interval $[0, a]$. Then the zeros of K_n are in $(0, a)$ and by (4.2)

$$\left|\frac{K'_n(0)}{K_n(0)}\right| > \frac{n}{a}.$$

B. Suppose that the weight function w is uniquely determined by the sequence of moments $\{c_n\}$. Suppose moreover that 0 is in the support of w. Then it follows from [4, p. 58–60] that the smallest zero x_1 of K_n tends to 0 if $n \to \infty$. Then (4.2) implies

$$\left|\frac{K'_n(0)}{K_n(0)}\right| \to \infty.$$

On the other hand, Koekoek [8] has given an example of a weight function for which $\{\alpha_n\}$ is bounded. Hence there exists a weight function which does not satisfy the condition $|K'_n(0)/K_n(0)| \to \infty$ if $n \to \infty$.

5. The Zeros of S_n: Introduction

Let the support of w be contained in the interval $[0, a)$, where a may be finite or infinite.

THEOREM 5.1. *The polynomial S_n has n real, simple zeros; at most one of them is outside* $(0, a)$. *If S_n has a zero outside* $(0, a)$, *then the zero is in* $(-\infty, 0]$.

Proof. Let $v_1, v_2, ..., v_k$ denote the zeros of S_n in $(0, a)$ of odd multiplicity. Put

$$\varphi(x) = (x - v_1)(x - v_2) \cdots (x - v_k). \tag{5.1}$$

Remark that for $k \geqslant 1$, $\varphi(0)$, and $\varphi'(0)$ have opposite sign. Then $\varphi(x)\, S_n(x)$ does not change sign on $[0, a)$. Suppose degree $\varphi \leqslant n-1$. Then $\langle \varphi, S_n \rangle = 0$, i.e.,

$$\int_0^\infty w(x)\, \varphi(x)\, S_n(x)\, dx + M\varphi(0)\, S_n(0) + N\varphi'(0)\, S_n'(0) = 0. \tag{5.2}$$

If $N = 0$ this is obviously impossible. Hence for $N = 0$ we have degree $\varphi = n$. If $N > 0$ relation (5.1) does not lead to a contradiction. Suppose now that, for $N > 0$, degree $\varphi < n - 2$. Then also $\langle x\varphi, S_n \rangle = 0$, i.e.,

$$\int_0^\infty w(x)\, x\varphi(x)\, S_n(x)\, dx + N\varphi(0)\, S_n'(0) = 0. \tag{5.3}$$

It follows from (5.2) and (5.3) that $\varphi'(0)\, S_n'(0)$ and $\varphi(0)\, S_n'(0)$ should have the same sign. This is a contradiction. Hence, for $N > 0$, degree $\varphi = n$ or $n - 1$ and the first part of the theorem follows.

Suppose now that degree $\varphi = n - 1$. Then there is one zero v_n of S_n outside $(0, a)$. Relation (5.2) still holds. If $S_n(0) = 0$ then $v_n = 0$ is the last zero of S_n. If $S_n(0) \neq 0$ then (5.2) implies that $\varphi(0)\, S_n(0)$ and $\varphi'(0)\, S_n'(0)$ should have opposite sign. Since $\varphi(0)$ and $\varphi'(0)$ have opposite sign, we conclude that $S_n(0)$ and $S_n'(0)$ have the same sign. But then the last zero v_n cannot lie in $[a, \infty)$. Hence $v_n \in (-\infty, 0)$.

COROLLARY 5.1. *Concerning the position of the zeros of S_n there are two different possible situations.*

1. *All n zeros $\xi_1 < \xi_2 < \cdots < \xi_n$ lie in* $(0, a) \subset (0, \infty)$. *Since the leading coefficient $\tilde{a}_n$ of S_n is positive, then* $\operatorname{sgn} S_n(0) = (-1)^n$, $\operatorname{sgn} S_n'(0) = (-1)^{n-1}$.

2. *There are $n - 1$ zeros $\xi_2 < \xi_3 < \cdots < \xi_n$ of S_n in* $(0, a) \subset (0, \infty)$ *and there is one zero $\xi_1 \in (-\infty, 0]$. In this case we write $\xi_1 = -\rho$, $\rho \geqslant 0$. Then*

$$S_n(x) = \tilde{a}_n(x + \rho)(x - \xi_2) \cdots (x - \xi_n). \tag{5.4}$$

If $\rho \neq 0$, then $\operatorname{sgn} S_n(0) = (-1)^{n-1}$. *It is stated in the proof of Theorem* 5.1

that in this case $S_n(0)$ *and* $S_n'(0)$ *should have the same sign. Hence* $\operatorname{sgn} S_n'(0) = (-1)^{n-1}$.

It follows from Remark 3.2, *recall* (2.3): $\operatorname{sgn} K_n(0) = (-1)^n$, *that the first situation occurs when* $0 \leqslant \alpha_n N < 1$ *and the second one when* $\alpha_n N \geqslant 1$.

THEOREM 5.2. *If* S_n *has a zero in* $(-\infty, 0]$ *for* $n = n_0$, *then* S_n *has a zero in* $(-\infty, 0)$ *for all* n *with* $n > n_0$.

Proof. This statement is a direct consequence of Corollaries 4.1 and 5.1.

Finally we derive some simple estimates for the negative zero $-\rho$ of S_n.

THEOREM 5.3. *Suppose that* S_n *has a zero* $-\rho \in (-\infty, 0)$. *Then*

(a) $\rho < \xi_2$, *where* ξ_2 *denotes the smallest positive zero of* S_n,

(b) *if the support of* w *is contained in the finite interval* $[0, a)$, *then* $\rho < a/(n-1)$,

(c) *if* $M \neq 0$, *then* $\rho < \frac{1}{2}\sqrt{N/M}$.

Proof. Corollary 5.1.2 implies

$$0 < \frac{S_n'(0)}{S_n(0)} = \frac{1}{\rho} - \frac{1}{\xi_2} - \frac{1}{\xi_3} \cdots - \frac{1}{\xi_n}.$$

Hence

$$\frac{1}{\rho} > \frac{1}{\xi_2} + \frac{1}{\xi_3} + \cdots + \frac{1}{\xi_n}. \tag{5.5}$$

Then $1/\rho > 1/\xi_2$, i.e., $\rho < \xi_2$.

If a is finite, then $\xi_2 < \xi_3 < \cdots < \xi_n < a$. Then (5.5) gives

$$\frac{1}{\rho} > \frac{n-1}{a}.$$

Hence $\rho < a/(n-1)$. In order to prove (c) we remark that by (5.1) and (5.4), S_n can be written as

$$S_n(x) = \tilde{a}_n(x + \rho)\, \varphi(x).$$

Then $\varphi(x)\, S_n(x)$ is non-negative on $[0, \infty)$ and (5.2) implies

$$M\varphi(0)\, S_n(0) + N\varphi'(0)\, S_n'(0) < 0$$

or

$$M\rho\varphi(0)^2 + N\varphi'(0)\{\varphi'(0) + \rho\varphi(0)\} < 0.$$

We obtain

$$\rho\{M\varphi(0)^2+N\varphi'(0)^2\}<-N\varphi'(0)\,\varphi(0)=N\,|\varphi'(0)\,\varphi(0)|.$$

On the other hand,

$$M\varphi(0)^2+N\varphi'(0)^2\geqslant 2\sqrt{MN}\,|\varphi(0)\,\varphi'(0)|.$$

Hence $\rho<\frac{1}{2}\sqrt{N/M}$.

6. Location of the Zeros of S_n

The following observation is due to Christoffel (see [11, p. 30]). Put

$$x^2Q_{n-1}(x)=\begin{vmatrix} K_{n-1}(x) & K_n(x) & K_{n+1}(x) \\ K_{n-1}(0) & K_n(0) & K_{n+1}(0) \\ K'_{n-1}(0) & K'_n(0) & K'_{n+1}(0) \end{vmatrix} \qquad \text{for} \quad n\geqslant 1. \tag{6.1}$$

Then obviously Q_{n-1} is a polynomial of degree $n-1$ with leading coefficient, $a_{n+1}\{K_{n-1}(0)\,K'_n(0)-K_n(0)\,K'_{n-1}(0)\}\neq 0$ (compare (4.3)). Moreover

$$\int_0^\infty w(x)\,x^ix^2Q_{n-1}(x)\,dx=0 \qquad \text{for} \quad i=0,1,...,n-2.$$

Hence

$$Q_{n-1}(x)=\text{const}\,K^{(2)}_{n-1}(x). \tag{6.2}$$

Lemma 6.1. *Between two consecutive zeros of K_n there is exactly one zero of $K^{(2)}_{n-1}$.*

Proof. Using (6.1), (6.2), and the recurrence relation we may write

$$x^2K^{(2)}_{n-1}(x)=(d_1x+d_2)\,K_n(x)+d_3K_{n-1}(x)$$

for some constants d_1, d_2, d_3. Since $K_n(0)\neq 0$ we have $d_3\neq 0$. Let x_i and x_{i+1} denote two consecutive zeros of K_n. It is well known that K_{n-1} has exactly one zero in (x_i, x_{i+1}). Hence $K_{n-1}(x_i)$ and $K_{n-1}(x_{i+1})$ have opposite sign. Then also $K^{(2)}_{n-1}(x_i)$ and $K^{(2)}_{n-1}(x_{i+1})$ have opposite sign. This implies that $K^{(2)}_{n-1}$ has at least one zero in (x_i, x_{i+1}). Since this holds for $i=1,2,...,n-1$ we conclude that $K^{(2)}_{n-1}$ has exactly one zero in (x_i, x_{i+1}).

COROLLARY 6.1. *Since the zeros of K_n and $K^{(2)}_{n-1}$ mutually separate each other we can also state that between two consecutive zeros of $K^{(2)}_{n-1}$ there is exactly one zero of K_n. Reading Lemma 6.1 with the weight function w replaced by $x^2w(x)$ we obtain that between two consecutive zeros of $K^{(2)}_{n-1}$ there is exactly one zero of $K^{(4)}_{n-2}$.*

LEMMA 6.2. *Let $y_1 < y_2 < \cdots < y_{n-1}$ denote the zeros of $K^{(2)}_{n-1}$. Then $K_n(y_i)$ and $K^{(4)}_{n-2}(y_i)$ have opposite sign. The sign of $K^{(4)}_{n-2}(y_i)$ is $(-1)^{n-1-i}$.*

Proof. Let x_n denote the largest zero of K_n and z_{n-2} the largest zero of $K^{(4)}_{n-2}$. Lemma 6.1 implies $z_{n-2} < y_{n-1} < x_n$. Since all leading coefficients are positive we have

$$K_n(y_{n-1}) < 0 < K^{(4)}_{n-2}(y_{n-1})$$

and the lemma is proved for $i = n-1$.

Running from y_{n-1} to y_{n-2} we pass exactly one zero of K_n and exactly one zero of $K^{(4)}_{n-2}$. Then in y_{n-2} we conclude that $K_n(y_{n-2})$ and $K^{(4)}_{n-2}(y_{n-2})$ have again different sign. Moreover the sign of the latter is -1. Hence the lemma is proved for $i = n-2$. Proceeding in this way we prove the lemma for $i = n-1, n-2, n-3, ..., 1$.

The lemma enables us to give a complete description of the position of the zeros of S_n if S_n has a negative zero.

THEOREM 6.1. *Let $y_1 < y_2 < \cdots < y_{n-1}$ denote the zeros of $K^{(2)}_{n-1}$. Suppose that S_n has a zero in $(-\infty, 0]$. Then S_n has a zero in $(0, y_1)$ and a zero in every interval (y_i, y_{i+1}), $i = 1, 2, ..., n-2$. The non-positive zero lies in $(-y_1, 0]$.*

Proof. By Theorem 3.1 we have

$$S_n(y_i) = (1 - \alpha_n N)\, K_n(y_i) + B_3\, y_i^2 K^{(4)}_{n-2}(y_i),$$

where $B_3 > 0$ and $1 - \alpha_n N \leqslant 0$ (compare Corollary 5.1). Then Lemma 6.2 implies $\operatorname{sgn} S_n(y_i) = (-1)^{n-1-i}$. Hence every interval (y_i, y_{i+1}), $i = 1, 2, ..., n-2$ contains at least one zero of S_n.

Moreover Lemma 6.2 implies $\operatorname{sgn} S_n(y_1) = (-1)^n$. Suppose $S_n(0) \neq 0$. Then Corollary 5.1.2 says $\operatorname{sgn} S_n(0) = (-1)^{n-1}$ and there is at least one zero of S_n in $(0, y_1)$. If $S_n(0) = 0$ then, again by Corollary 5.1.2, we have $\operatorname{sgn} S_n'(0) = (-1)^{n-1}$ and again there has to be at least one zero of S_n in $(0, y_1)$. Since S_n has by assumption $n-1$ zeros in $(0, \infty)$ every interval $(0, y_1)$, (y_i, y_{i+1}), $i = 1, 2, ..., n-2$ has exactly one zero of S_n.

Finally let $-\rho$ denote the non-positive zero of S_n and ξ_2 the smallest positive zero of S_n. Then by Theorem 5.3(a), $\rho < \xi_2 < y_1$.

It is possible to represent S_n as a linear combination of $K_n(x)$, $xK_{n-1}^{(2)}(x)$, and $K_{n-1}^{(2)}(x)$. However, the coefficients are more complicated than those in Theorem 3.1.

THEOREM 6.2. *For $n \geq 0$ the polynomials $S_n^{M,N}$ can be written as*

$$S_n^{M,N}(x) = D_1 K_n(x) + D_2 x K_{n-1}^{(2)}(x) + D_3 K_{n-1}^{(2)}(x), \tag{6.3}$$

where

$$D_1 = 1 - \frac{N}{K_n(0)^2} \{K_n'(0) K_{n-1}^{(2)}(0) + K_n(0) K_{n-1}^{(2)\prime}(0)\} - MN \frac{K_{n-1}^{(2)}(0)^2}{K_n(0)^2},$$

$$D_2 = M + N \frac{K_n'(0)^2}{K_n(0)^2} + \frac{MN}{K_n(0)^2} \{K_n'(0) K_{n-1}^{(2)}(0) - K_n(0) K_{n-1}^{(2)\prime}(0)\},$$

$$D_3 = N \frac{K_n'(0)}{K_n(0)} + MN \frac{K_{n-1}^{(2)}(0)}{K_n(0)}.$$

Proof. We proceed as in the proof of Theorem 3.1. Obviously the righthand member of (6.3) is orthogonal with respect to the inner product (3.1) on x^i for $2 \leq i \leq n-1$ for every choice of D_1, D_2, and D_3. So we have to choose the coefficients in such a way that also $\langle 1, S_n \rangle = 0$ and $\langle x, S_n \rangle = 0$. This gives the equations

$$D_1 M K_n(0) - D_2 K_n(0) + D_3(K_n'(0) + M K_{n-1}^{(2)}(0)) = 0,$$
$$D_1 N K_n'(0) - D_2 N K_{n-1}^{(2)}(0) + D_3(-K_n(0) + N K_{n-1}^{(2)\prime}(0)) = 0.$$

From this system the coefficients can be derived.

Observe that the coefficients of N and MN in D_1 are positive, so D_1 may be zero for suitable choices of N and M. If $D_1 = 0$, then $S_n(x) = (D_2 x + D_3) K_{n-1}^{(2)}(x)$ and all zeros of $K_{n-1}^{(2)}(x)$ are zeros of S_n.

Finally we describe the behaviour of the zeros of $S_n = S_n^N$ for fixed n and M and variable N. Let, as before, $y_1 < y_2 < \cdots < y_{n-1}$ denote the zeros of $K_{n-1}^{(2)}$ and $\xi_1 < \xi_2 < \cdots < \xi_n$ those of S_n^N. If $N = 0$, then $S_n^N = K_n$ and by Lemma 6.1 the location of the zeros of S_n is as follows:

$$\xi_1 < y_1, \qquad \xi_n > y_{n-1}, \qquad \xi_{i+1} \in (y_i, y_{i+1}) \qquad \text{for} \quad i = 1, 2, ..., n-2. \tag{a}$$

On the other hand, if $N = N_1 > \alpha_n^{-1}$ then by Theorem 6.1 the location of the zeros is

$$\xi_1 < \xi_2 < y_1, \qquad \xi_{i+1} \in (y_{i-1}, y_i) \qquad \text{for} \quad i = 2, ..., n-1. \tag{b}$$

The zeros ξ_{i+1} are continuous functions of N. If N runs from $N = 0$ to

$N = N_1$ the situation (a) is continuously transformed in the situation (b). Hence ξ_{i+1} had to pass through y_i ($i = 1, 2, ..., n-1$). By Theorem 6.2,

$$S_n^N(y_i) = D_1 K_n(y_i), \qquad i = 1, 2, ..., n-1,$$

where $D_1 = D_1(N)$ is a linear function of N. (Recall that M and n are fixed.) Hence there is exactly one value $N_0 \in (0, N_1)$ such that $D_1(N_0) = 0$. For this value N_0 it follows $\xi_{i+1} = y_i$ for $i = 1, 2, ..., n-1$.

Now we may conclude that if $N \in [0, N_0)$, then the zeros of S_n^N are located according to position (a), if $N > N_0$, then the zeros of S_n^N are in position (b).

COROLLARY 6.2. *Either all zeros of $K_{n-1}^{(2)}$ are zeros of S_n, or between two consecutive zeros of $K_{n-1}^{(2)}$ there is exactly one zero of S_n.*

Remark 6.1. In general it is *not* true, that between two consecutive zeros of K_n, there is a zero of S_n. Let $x_1 < x_2 < \cdots < x_n$ denote the zeros of K_n. By Theorem 6.2,

$$S_n(x_i) = (D_2 x_i + D_3)\, K_{n-1}^{(2)}(x_i), \qquad i = 1, 2, ..., n. \tag{6.4}$$

Take $N = M \to \infty$, then $-D_3/D_2$ converges to

$$\tau = \frac{K_{n-1}^{(2)}(0)\, K_n(0)}{K_n(0)\, K_{n-1}^{(2)\prime}(0) - K_n'(0)\, K_{n-1}^{(2)}(0)}.$$

Now

$$\frac{1}{\tau} = \frac{K_{n-1}^{(2)\prime}(0)}{K_{n-1}^{(2)}(0)} - \frac{K_n'(0)}{K_n(0)} = \sum_{i=1}^{n} \frac{1}{x_i} - \sum_{i=1}^{n-1} \frac{1}{y_i}.$$

By Lemma 6.1, $x_i < y_i < x_{i+1}$, $i = 1, 2, ..., n-1$. Hence

$$\frac{1}{x_n} < \frac{1}{\tau} < \frac{1}{x_1},$$

i.e., $x_1 < \tau < x_n$.

This implies that we can choose N and M in such a way that the zero T of $D_1 x + D_3$ is between two consecutive zeros, say x_j and x_{j+1} of K_n. Let moreover N be chosen so large that S_n has a negative zero. If $i \in \{1, 2, ..., n-1\}$, $i \neq j$, then by (6.4) and Lemma 6.1, $S_n(x_i)$ and $S_n(x_{i+1})$ have opposite sign. Hence S_n has at least $n-2$ zeros in $(x_1, x_n) \backslash (x_j, x_{j+1})$.

Since S_n also has a negative zero, S_n has at least $n-1$ zeros outside (x_j, x_{j+1}). However, by (6.4), $S_n(x_j)$ and $S_n(x_{j+1})$ have the same sign, so the last zero of S_n cannot lie in (x_j, x_{j+1}).

Remark 6.2. Several other representations of S_n in standard polynomials can be derived. Obviously we can write

$$S_n(x) = \sum_{i=0}^{n} A_i K_i^{(2)}(x).$$

Then, for $i \in \{0, 1, ..., n\}$,

$$A_i \int_0^\infty w(x)\, x^2 K_i^{(2)}(x)^2\, dx = \int_0^\infty w(x)\, x^2 S_n(x)\, K_i^{(2)}(x)\, dx = \langle S_n, x^2 K_i^{(2)}(x) \rangle.$$

The last member is zero for $i \leqslant n-3$. Hence

$$S_n(x) = A_n K_n^{(2)}(x) + A_{n-1} K_{n-1}^{(2)}(x) + A_{n-2} K_{n-2}^{(2)}(x),$$

where the coefficients A_n, A_{n-1}, A_{n-2} depend on n, N, and M. This means that S_n is quasi-orthogonal with respect to $\{K_n^{(2)}(x)\}$ of order 2. Marcellan and Ronveaux [10] have proved that $x^2 S_n(x)$ is quasi-orthogonal with respect to $\{K_n\}$ of order 4.

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